Beta Distribution of Ist Kind

Let 'X' be a continuous random variable with interval (0, 1) is said to be Beta distribution of 1st kind, having p.d.f:

$$f(x) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1} \qquad 0 \le x \le 1$$

And its function is:

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

It has two parameters a & b.

Properties

i) Beta distribution is a continuous distribution.
ii) The total area under the curve is unity.
iii) The range of the distribution is 0 to 1.
iv) It has two parameters a & b.

v) The mean of the beta distribution of first kind is E(x)= $\frac{a}{a+b}$.

vi) The variance of the beta distribution of first kind is Var(x)= $\frac{ab}{(a+b)^2(a+b+1)}$.

Prove that total area under the curve is unity

Proof:
Let by definition:

Total Area: Area= $\int f(x)dx$

As x$\approx$ beta 1st (a,b)

$$f(x) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1} \qquad 0 \le x \le 1$$

$$\text{Area} = \int_0^1 \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1}dx$$

$$= \frac{1}{\beta(a,b)} \int_0^1 x^{a-1}(1-x)^{b-1}dx \qquad \text{(A)}$$

As we know that beta function is

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx \qquad \text{(B)}$$

Comparing (A) & (B)
a = a & b = b

$$\beta(a,b) = \beta(a,b)$$

Put in (A)

$$\text{Area} = \frac{1}{\beta(a,b)}\beta(a,b) = 1$$

Hence Proved

Derive rth moment about origin and use it to find mean & variance

Solution:
Let by definition

$$\mu_r' = E(x^r) = \int x^r f(x)dx$$

As x$\approx$ beta 1st (a,b)

$$f(x) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1} \qquad 0 \le x \le 1$$

$$\mu_r' = \frac{1}{\beta(a,b)}\int_0^1 x^r x^{a-1}(1-x)^{b-1}dx$$

$$\mu_r' = \frac{1}{\beta(a,b)}\int_0^1 x^{(r+a)-1}(1-x)^{b-1}dx \qquad \text{(A)}$$

As we know that beta function is

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx \qquad \text{(B)}$$

Comparing (A) & (B) and we get
a = a+r & b = b

$$\beta(a,b) = \beta(a+r,b)$$

Put in (A)

$$\mu_r' = \frac{1}{\beta(a,b)}\beta(a+r,b)$$

$$\mu_r' = \frac{\dfrac{\Gamma(r+a)\Gamma(b)}{\Gamma(r+a+b)}}{\dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}}$$

$$\mu_r' = \frac{\Gamma(r+a)\Gamma(b)}{\Gamma(r+a+b)}.\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

$$\mu_r' = \frac{\Gamma(r+a)\Gamma(a+b)}{\Gamma(r+a+b)\Gamma(a)} \qquad ©$$

Hence the required result

Mean = $\mu_1' = E(x)$

Put r = 1 in eq (C)

$$\mu_1' = \frac{\Gamma(1+a)}{\Gamma(1+a+b)}.\frac{\Gamma(a+b)}{\Gamma(a)}$$

$$\mu_1' = \frac{a\Gamma(a)}{(a+b)\Gamma(a+b)}.\frac{\Gamma(a+b)}{\Gamma(a)}$$

$$\mu_1' = \frac{a}{(a+b)}$$

$$\mu_1' = \frac{a}{(a+b)}. = Mean = A.M$$

Now, put r = 2 in eq.(C)

$$\mu_2' = \frac{\Gamma(2+a)}{\Gamma(2+a+b)}.\frac{\Gamma(a+b)}{\Gamma(a)}$$

$$\mu_2' = \frac{(a+1)\overline{)a+1}}{(a+b+1)\overline{)a+b+1}} . \frac{\overline{)a+b}}{\overline{)a}}$$

$$\mu_2' = \frac{a(a+1)\overline{)a}}{(a+b)(a+b+1)\overline{)a+b}} . \frac{\overline{)a+b}}{\overline{)a}}$$

$$\mu_2' = \frac{a(a+1)}{(a+b)(a+b+1)}.$$

$$Var(X) = \mu_2 = \mu_2' - (\mu_1')^2 = E(X^2) - (E(X))^2$$

$$Var(x) = \frac{a(a+1)}{(a+b)(a+b+1)} - \left(\frac{a}{a+b}\right)^2$$

$$\text{Var(X)} = \frac{a}{a+b}\left[\frac{(a+1)}{(a+b+1)} - \frac{a}{a+b}\right]$$

$$\text{Var(X)} = \frac{a}{a+b}\left[\frac{(a+1)(a+b)}{(a+b+1)} - \frac{a(a+b+1)}{a+b}\right]$$

$$\text{Var(X)} = \frac{a}{a+b}\left[\frac{a^2+a+ab+b-a^2-ab-a}{(a+b+1)(a+b)}\right]$$

$$Var(x) = \frac{ab}{(a+b)^2(a+b+1)}$$

Find mode of beta distribution of kind 1st

Proof:

As x ≈ beta 1st (a,b)

$$f(x) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1} \qquad 0 \le x \le 1$$

If following two conditions are satisfied then mode exists.

$$f(x') = 0 \qquad \text{or} \qquad \frac{d}{dx}\log f(x) = 0$$

$$f(x') < 0 \qquad \text{or} \qquad \frac{d^2}{dx^2}\log f(x) < 0$$

$$f(x) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1}$$

Taking log on both sides:

$$\log f(x) = \log\left[\frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1}\right]$$

$$\log f(x) = \log 1 - \log\beta(a,b) + (a-1)\log x + (b-1)\log(1-x)$$

Differentiate w.r.t to ‘x’

$$\frac{d}{dx}\log f(x) = \frac{d}{dx}\left[\log 1 - \log\beta(a,b) + (a-1)\log x + (b-1)\log(1-x)\right]$$

$$\frac{d}{dx}\log f(x) = 0 - 0 + \frac{(a-1)}{x} + \frac{(b-1)}{(1-x)}(-1)$$

$$\frac{d}{dx}\log f(x) = \frac{(a-1)}{x} - \frac{(b-1)}{(1-x)} \qquad \text{(A)}$$

$$0=\frac{(a-1)(1-x)-x(b-1)}{x(1-x)}$$

$$x(b-1)=(a-1)(1-x)$$

$$xb-x=a-ax-1+x$$

$$xb-x+ax-x=a-1$$

$$x(b+a-2)=a-1$$

$$x=\frac{a-1}{(b+a-2)}$$

Again diff. eq(A) w.r.t to ‘x’

$$\frac{d^2}{dx^2}\log f(x)=-\frac{(a-1)}{x^2}+\frac{(b-1)}{(1-x)^2}$$

Put $x=\dfrac{a-1}{(b+a-2)}$

$$\frac{d^2}{dx^2}\log f(x)=-\frac{(a-1)}{\left(\frac{a-1}{b+a-2}\right)^2}+\frac{(b-1)}{(1-\frac{a-1}{b+a-2})^2}$$

$$\frac{d^2}{dx^2}\log f(x)=-\frac{(a-1)(b+a-2)^2}{(a-1)^2}+\frac{(b-1)}{(\frac{b+a-2-a-1}{b+a-2})^2}$$

$$\frac{d^2}{dx^2}\log f(x)=-\frac{(a-1)(b+a-2)^2}{(a-1)^2}+\frac{(b-1)(b+a-2)^2}{(b-1)^2}$$

$$\frac{d^2}{dx^2}\log f(x)=-\frac{(b+a-2)^2}{(a-1)}+\frac{(b+a-2)^2}{(b-1)}$$

$$\frac{d^2}{dx^2}\log f(x)=-(b+a-2)^2\left[\frac{1}{(a-1)}-\frac{1}{(b-1)}\right]$$

$$\frac{d^2}{dx^2}\log f(x)=-(b+a-2)^2\left[\frac{b-1+a+1}{(a-1)(b-1)}\right]$$

$$\frac{d^2}{dx^2}\log f(x)=-(b+a-2)^2\left[\frac{b+a}{(a-1)(b-1)}\right]<0$$

Hence both conditions are satisfied. So mode exists. The mode of beta distribution of kind 1st is

$$\hat{x}=\frac{a-1}{(b+a-2)}.$$

Find the harmonic mean of beta distribution of kind 1st

Proof:

Let by definition

$$\text{H.M} = 1\Big/E\left(\frac{1}{x}\right) \qquad \text{(A)}$$

$$E\left(\frac{1}{x}\right)=\int\frac{1}{x}f(x)dx$$

As x $\approx$ beta 1st (a,b)

$$f(x)=\frac{1}{\beta(a,b)}x^{a-1}(1-x)^{b-1} \qquad 0\le x\le 1$$

$$E\left(\frac{1}{x}\right)=\frac{1}{\beta(a.b)}\int_0^1 \frac{1}{x}x^{a-1}(1-x)^{b-1}dx$$

$$E\left(\frac{1}{x}\right)=\frac{1}{\beta(a,b)}\int_0^1 x^{(a-1)-1}(1-x)^{b-1}dx \qquad \text{(B)}$$

As we know that beta function is

$$\beta(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx \qquad \text{(C)}$$

Comparing (B) & (C) and we get

a = a-1 & b = b

$$\beta(a,b)=\beta(a-1,b)$$

Put in (B)

$$E\left(\frac{1}{x}\right)=\frac{\beta(a-1,b)}{\beta(a,b)}$$

$$E\left(\frac{1}{x}\right)=\frac{1}{\dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}}\frac{\Gamma(a-1)\Gamma(b)}{\Gamma(a-1+b)}$$

$$E\left(\frac{1}{x}\right)=\frac{\Gamma(a+b)}{\Gamma(a)}\frac{\Gamma(a-1)}{\Gamma(a-1+b)}$$

$$E\left(\frac{1}{x}\right)=\frac{\Gamma(a+b)}{(a-1)\Gamma(a-1)}\frac{\Gamma(a-1)}{\Gamma(a-1+b)}$$

$$E\left(\frac{1}{x}\right)=\frac{(a+b-1)\Gamma(a+b-1)}{(a-1)\Gamma(a-1+b)}$$

$$E\left(\frac{1}{x}\right)=\frac{(a+b-1)}{(a-1)}$$

Put in (A)

$$\text{H.M} = \frac{1}{\dfrac{(a+b-1)}{(a-1)}}=\frac{(a-1)}{(a+b-1)}$$

$$\text{H.M} = \frac{(a-1)}{(a+b-1)}$$

Hence required result.

Q. If 'x' follows beta distribution of kind first with parameter (m,n) then show that $\beta(m,n)=1$

Solution: Given that

As x $\approx$ beta 1st (m,n)

$$f(x)=\frac{1}{\beta(m,n)}x^{m-1}(1-x)^{n-1}$$

Then beta function is:

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$$ Put m = n = 1

$$\beta(m,n)=\int_0^1 x^{1-1}(1-x)^{1-1}dx$$

$$\beta(m,n)=\int_0^1 1dx$$

$$\beta(m,n)=x\Big|_0^1$$

$$\beta(m,n)=1-0=1$$ Hence prove.

Q. If 'm = n =1 then beta distribution of kind first becomes rectangular distribution

Solution: Given that

As x ≈ beta 1st (m,n)

$$f(x)=\frac{1}{\beta(m,n)}x^{m-1}(1-x)^{n-1}$$

m = n = 1

$$f(x)=\frac{1}{\beta(1,1)}x^{1-1}(1-x)^{1-1}$$

$$f(x)=\frac{1}{\beta(1,1)}x^{0}(1-x)^{0}$$

$$f(x)=\frac{1}{\frac{\overline{)1}\,\overline{)1}}{\overline{)1+1}}}$$

$$f(x)=\frac{\overline{)2}}{\overline{)1}\,\overline{)1}}$$

$$f(x)=(2-1)!=1!=1$$

Hence, beta distribution of 1st kind approaches to rectangular distribution. When m=n =1

Q. Show that $\beta(m,n)=\beta(n,m)$

Solution:

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx \qquad \text{(i)}$$

$Put \quad z=1-x, \qquad x=1-z, \qquad dx=-dz$ While limits will be

$As \;\; x\to 0 \;\; then \;\; z\to 1 \qquad \& \qquad As \;\; x\to 1 \;\; then \;\; z\to 0$

$$\beta(m,n)=\int_1^0 (1-z)^{m-1}z^{n-1}(-dz)=\int_0^1 z^{n-1}(1-z)^{m-1}dz=\beta(n,m)$$ Hence prove

Q. Show that $\beta(m,n)=2\int_0^{\frac{\pi}{2}}(Sin\;\;\theta)^{2m-1}(Cos\;\;\theta)^{2n-1}d\theta$

Solution:

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx \qquad \text{(i)}$$

Let $x=Sin^2\;\theta, \qquad dx=2Sin\;\theta\left(Cos\;\;\theta\right)d\theta, \qquad 1-x=1-Sin^2\;\theta=Cos^2\;\theta$

While the limits will be

$$As\ \ x \to 1 \quad then \quad \theta \to \frac{\pi}{2} \qquad \& \qquad As\ \ x \to 0 \quad then \quad \theta \to 0$$

Then (i) becomes

$$\beta(m,n) = \int_0^{\frac{\pi}{2}} \left(Sin^2\theta\right)^{m-1} (Cos^2\theta)^{n-1} 2Sin\ \theta Cos\ \ \theta\, d\theta = 2\int_0^{\frac{\pi}{2}} \left(Sin\theta\right)^{2m-2} (Cos\theta)^{2n-2} Sin\ \theta Cos\ \ \theta\, d\theta$$

$$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \left(Sin\theta\right)^{2m-1} (Cos\theta)^{2n-1}\, d\theta$$ Hence proved

Prove that $\beta(\frac{1}{2},\frac{1}{2}) = \pi$

Solution; As we know that

$$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \left(Sin\theta\right)^{2m-1} (Cos\theta)^{2n-1}\, d\theta$$ Put $m = n = \frac{1}{2}$

$$\beta(\frac{1}{2},\frac{1}{2}) = 2\int_0^{\frac{\pi}{2}} \left(Sin\theta\right)^{2\frac{1}{2}-1} (Cos\theta)^{2\frac{1}{2}-1}\, d\theta$$

$$\beta(\frac{1}{2},\frac{1}{2}) = 2\int_0^{\frac{\pi}{2}} \left(Sin\theta\right)^{0} (Cos\theta)^{0}\, d\theta$$

$$\beta(\frac{1}{2},\frac{1}{2}) = 2\int_0^{\frac{\pi}{2}} d\theta$$

$$\beta(\frac{1}{2},\frac{1}{2}) = 2\theta\Big|_0^{\frac{\pi}{2}} = 2.\frac{\pi}{2} - 0 = \pi$$

Derive moment generating function of beta distribution of 1st kind

Let by definition of m.g.f of beta distribution of 1st kind

$$M_0(t) = E(e^{tx}) = \int e^{tx} f(x)dx$$

As we know that

$$e^{tx} = 1 + \frac{(\text{tx})^1}{1!} + \frac{(\text{tx})^2}{2!} + \frac{(\text{tx})^3}{3!} + \ldots$$

$$e^{tx} = \sum_{r=0}^{\infty} \frac{(\text{tx})^r}{\text{r}!}$$

$$e^{tx} = \sum_{r=0}^{\infty} \frac{\text{t}^r \text{x}^r}{\text{r}!}$$

Then we get

$$M_0(t) = \int \sum_{r=0}^{\infty} \frac{\text{t}^r \text{x}^r}{\text{r}!} f(x)dx$$

$$M_0(t) = \sum_{r=0}^{\infty} \frac{\text{t}^r}{\text{r}!} \int \text{x}^r\ f(x)dx \qquad \text{(A)}$$

Now we consider

$$\int \text{x}^r\ f(x)dx$$

As x ≈ beta 1st (a,b)

$$f(x)=\frac{1}{\beta(a,b)}x^{a-1}(1-x)^{b-1} \qquad 0\le x\le 1$$

$$\int \mathrm{x}^r\ f(x)dx=\frac{1}{\beta(a,b)}\int_0^1 x^r x^{a-1}(1-x)^{b-1}dx$$

$$\int \mathrm{x}^r\ f(x)dx=\frac{1}{\beta(a,b)}\int_0^1 x^{(r+a)-1}(1-x)^{b-1}dx \qquad \text{(B)}$$

As we know that beta function is

$$\beta(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx \qquad \text{(C)}$$

Comparing (B) & (C) and we get

a = a+r & b = b

$$\beta(a,b)=\beta(a+r,b)$$

Put in (B)

$$\int \mathrm{x}^r\ f(x)dx=\frac{1}{\beta(a,b)}\beta(a+r,b)$$

$$\int \mathrm{x}^r\ f(x)dx=\frac{\Gamma(r+a)\Gamma(b)\,/\,\Gamma(r+a+b)}{\Gamma(a)\Gamma(b)\,/\,\Gamma(a+b)}$$

$$\int \mathrm{x}^r\ f(x)dx=\frac{\Gamma(r+a)\Gamma(b)}{\Gamma(r+a+b)}\cdot\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

$$\int \mathrm{x}^r\ f(x)dx=\frac{\Gamma(r+a)}{\Gamma(r+a+b)}\cdot\frac{\Gamma(a+b)}{\Gamma(a)}$$

$$\int \mathrm{x}^r\ f(x)dx=\frac{\Gamma(r+a)\Gamma(a+b)}{\Gamma(r+a+b)\Gamma(a)}$$

Pun in (A)

$$M_0(t)=\sum_{r=0}^{\infty}\frac{\mathrm{t}^r}{\mathrm{r!}}\frac{\Gamma(r+a)\Gamma(a+b)}{\Gamma(r+a+b)\Gamma(a)} \qquad \text{Required m.g.f} \qquad \text{(D)}$$

As we know the relationship b/w m.g.f and rth moment about origin

$$M_0(t)=\sum_{r=0}^{\infty}\frac{\mathrm{t}^r}{\mathrm{r!}}u'_r \qquad \text{(E)}$$

Comparing (D) and (E) we get

$$u'_r=\frac{\Gamma(r+a)\Gamma(a+b)}{\Gamma(r+a+b)\Gamma(a)}$$

Prepared by: Muhammad Riaz
M.Sc: Statistics (Silver medalist)
I.U.B (2006-08)
Lecturer: Statistics
NICAAS College (RYK)
03337368518